MALLIAVIN DIFFERENTIABILITY OF TIME-ADVANCED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

OLIVIER MENOUKEU-PAMEN

Abstract. We consider a time-advanced backward stochastic differential equations (ABSDEs). We study Malliavin differentiability of solutions of such equations and derive equations satisfied by the Malliavin derivative processes.

1. Introduction

Backward stochastic differential equations (BSDEs) was first introduced by Bismut [1] in the linear case and extended by Pardoux and Peng [10] in the non-linear case. This theory has been extensively studied in the last decade. One of the most important feature of BSDEs is that they can be applied to mathematical finance and control theory.

El Karoui et al. [3] studied the properties of differentiation on Wiener space of the solution of a BSDE in the spirit of Pardoux and Peng [10]. Applying these results in finance, they showed that the portfolio process of a hedging strategy corresponds to the Malliavin derivative of the price process.

In this paper, we shall study variational smoothness of solutions of time-advanced BSDEs. More exactly, we study Malliavin differentiability of BSDEs with dynamics satisfying:

$$\begin{aligned} dY(t) &= E\Big[b(t, Y(t), Y(t+\delta), Y_t, Z(t), Z(t+\delta), Z_t)\Big|\mathcal{F}_t\Big]\,\mathrm{d}t + Z(t)\,\mathrm{d}B(t),\ \ t\in[0,T] \\ Y(t) &= G(t),\ \ Z(t) = G_1(t),\ \ t\in[T, T+\delta]. \end{aligned}$$

Note in this equation that, the generator b at time t depends on the future value of the solution $(Y(s), Z(s)),\ \ t \le s \le t+\delta$.

Existence and uniqueness of solutions of time-advanced BSDEs was first studied by Peng and Yang [7] in the continuous case, and generalized by Øksendal et al. [8] in the jumps case.

In the present paper, we first establish a $\mathbb{L}^p$ approximation of solution of time-advanced BSDEs with a Lipschitz generator. Secondly, we prove Malliavin differentiability of the

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soluition of time-advanced BSDE and show that the Malliavin derivative of the solution of time-advanced BSDE is solution of a linear time-advanced BSDE. We also show that the connection between Z and the Malliavin trace of Y is still valid in the time-advanced case.

The paper is organized as follows: In Section 2, we give the framework needed to establish our results. Section 3 contains the main results of the paper, while Section 4 is devoted to the proof of these results.

2. Notations and Assumptions

In this section, we introduce some spaces and give conditions under which we will derive our main result.

Denote by $(\Omega, \mathcal{F}, P)$ a complete probability space. Let $\{\mathcal{F}_t\}_{0\leq t\leq T}$ be the natural filtration generated by the Brownian motion $B = \{B(t), t \in [0,T]\}$ where $\mathcal{F}_0$ contains all P null sets of $\mathcal{F}$. It is a complete right continuous filtration. From now on, we fix the final time $T > 0$.

Now, we define the following spaces of processes.

Definition 2.1. For $p \geq 2,\ \delta > 0$,

- $L^p_T(\mathbb{R})$ is the space of all $\mathcal{F}_T$–measurable random variables $X : \Omega \to \mathbb{R}$ such that
$$\mathbb{E}[|X|^p] < \infty,$$
- $S^p_{T+\delta}(\mathbb{R}) = S^p(0, T+\delta; \mathbb{R})$, is the space of real valued *càdlàg* processes $(X(t))_{t\in[0,T+\delta]}$ measurable and satisfying:
  (1) $X(t)$ is $\mathcal{F}_t$-measurable a.e. in t where $\mathcal{F}_t = \mathcal{F}_T,\ t \in [T, T+\delta]$.
  (2) $E\Big[\sup_{0\leq t\leq T+\delta} |X(t)|^p\Big] < \infty$.
- $H^p_{T+\delta}(\mathbb{R}) = H^p(0, T+\delta; \mathbb{R})$ is the space of real valued *càdlàg* processes $(X(t))_{t\in[0,T+\delta]}$ measurable and satisfying:
  (1) $X(t)$ is $\mathcal{F}_t$-measurable a.e. in t where $\mathcal{F}_t = \mathcal{F}_T,\ t \in [T, T+\delta]$.
  (2) $E\Big[\Big(\int_0^{T+\delta} |X(t)|^2\, \mathrm{d}t\Big)^{p/2}\Big] < \infty$.
- we define $\mathcal{V}^p_T(\mathbb{R}) := S^p_T(\mathbb{R}) \times H^p_T(\mathbb{R})$.

Given a stochastic process $X(t) \in H^p(0, T+\delta; \mathbb{R}) \cap S^p(0, T+\delta; \mathbb{R})$, we denoted by X_t, the $L^p(0, \delta; \mathbb{R}) \cap S^p(0, \delta; \mathbb{R})$-valued stochastic process by setting
$$X_t(s)(\omega) = X(t+s)(\omega);\ \ s \in [0, \delta].$$

Define by $V^p_\delta := L^p([0,\delta], \mathrm{d}s)$

Consider the following time-advanced BSDE

$$dY(t) = E\Big[b(t, Y(t), Y(t+\delta), Y_t, Z(t), Z(t+\delta), Z_t)\Big|\mathcal{F}_t\Big]\mathrm{d}t + Z(t)\,\mathrm{d}B(t),\ \ t \in [0,T] \quad (2.1)$$
$$Y(t) = G(t),\ \ Z(t) = G_1(t),\ \ t \in [T, T+\delta], \quad (2.2)$$

where $G(t,\omega)$ is a continuous real valued $\mathcal{F}_t$-measurable process, $G_1(t,\omega)$ is a continuous real valued $\mathcal{F}_t$-measurable process and $b : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ satisfying the following conditions:

Assumption A1.

(b1) $E\Big[\Big(\int_0^T |b(t,0,0,0,0,0,0)|^2 dt\Big)^{p/2}\Big] < \infty.$

(b2) Lipschitz condition: There exists a C such that

$$\begin{aligned}&|b(t,y,y_1,y_2,z,z_1,z_2) - b(t,\bar{y},\bar{y}_1,\bar{y}_2,\bar{z},\bar{z}_1,\bar{z}_2)|_H \\ &\leq C\Big(|y-\bar{y}| + |y_1-\bar{y}_1| + |y_2-\bar{y}_2|_{V_\delta^p} + |z-\bar{z}| + |z_1-\bar{z}_1| + |z_2-\bar{z}_2|_{V_\delta^p}\Big)\end{aligned}$$

We also assume that the terminal conditions are such that:

Assumption A2.

(g1) $G \in S^P(T,T+\delta;\mathbb{R}) \cap H^P(T,T+\delta;\mathbb{R}),\ \ G_1 \in H^P(T,T+\delta;\mathbb{R})$

2.1. **Differentiability on Wiener space.** In this Section we briefly recall some basic properties of Malliavin calculus related to this paper. We refer to [2, 6] for more information.

Denote by $B = \{B(h), h \in L^2([0,T];\mathbb{R})\}$ an isonormal Gaussian process associated with the space $L^2([0,T];\mathbb{R})$. We assume that B is defined on a complete probability space $(\Omega,\mathcal{F},P)$, and that $\mathcal{F}$ is generated by W. We shall also introduce the following spaces

- $C_p^\infty(\mathbb{R}^n)$ is the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth.
- $C_b^\infty(\mathbb{R}^n)$ is the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives are bounded.
- $\mathcal{S}$ is the class of smooth random variables F of the form
$$F = f(W(h^1),\dots,W(h^n)), \tag{2.3}$$
where $f \in C_p^\infty(\mathbb{R}^n)$ and $h^1,\dots,h^n \in L^2([0,T];\mathbb{R})$
- $\mathcal{S}_b$ is the class of smooth random variables F of the form such (2.3) with $f \in C_b^\infty(\mathbb{R}^n)$

Definition 2.2. Let $F \in \mathcal{S}$ of the form (2.3). Then the derivative of F is given by
$$D_tF = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(W(h^1),\dots,W(h^n))h^i(t). \tag{2.4}$$

For $p \geq 2$, it can be shown (see [6]) that D has a closed extension to the closure $\mathbb{D}^{1,p}$ of $\mathcal{S}_b$ with respect to the norm
$$\|F\|_{1,p} = \Big(E\Big[|F|^p + \Big(\int_0^T |D_tF|^2\mathrm{d}t\Big)^{p/2}\Big]\Big)^{1/p}.$$

Note that for $p = 2$, $\mathbb{D}^{1,2}$ is a Hilbert space. If F is $\mathcal{F}_s$-measurable, then $D_tF = 0$ for $t \in (s,T]$. We also have the following chain rules formula

Lemma 2.1. *Let $p \geq 2$ and $g \in C^1(\mathbb{R}^n)$ (the set of continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ with bounded partial derivatives). Assume that $F = (F^1, \dots, F^n)$ is a random vector with components belonging to the space $\mathbb{D}^{1,p}$. Then $g \circ F$ belongs to $\mathbb{D}^{1,p}$ and we have*

$$D(g \circ F) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} g(F) DF^i. \tag{2.5}$$

Proof. See [6]. □

For $p \geq 2$, let $\mathbb{L}^{1,p}(\mathbb{R}^n)$ be the set of $\mathbb{R}^n$-valued progressively measurable processes $u = \{u(t,\omega),\ 0 \leq t \leq T;\ \omega \in \Omega\}$ satisfying:

- u is adapted and $\Big(\int_0^T |u(t)|^2\,\mathrm{d}t\Big)^{1/2} \in L^p(\Omega \times [0,T])$,
- For λ-a.a. $t \in [0,T]$, $u(t) \in \mathbb{D}^{1,p}$,
- For some measurable version of $(s,t) \mapsto D_s u(t)$ we have $E\Big[\Big(\int_0^T \int_0^T |D_s u(t)|^2\,\mathrm{d}s\,\mathrm{d}t\Big)^{p/2}\Big] < \infty$

For $u \in \mathbb{L}^{1,p}(\mathbb{R}^n)$ we have

$$\|u\|_{\mathbb{L}^{1,p}} = E\Big[\Big(\int_0^T |u(t)|^2\,\mathrm{d}t\Big)^{p/2} + \Big(\int_0^T \int_0^T |D_s u(t)|^2\,\mathrm{d}s\,\mathrm{d}t\Big)^{p/2}\Big].$$

If we define by $\|Du\|^2 := \int_0^T \int_0^T |D_s u(t)|^2\,\mathrm{d}s\,\mathrm{d}t$. Then we have using Jensen's inequality

$$E\Big[\|Du\|^2\Big]^{p/2} \leq T^{p/2-1} \int_0^T |D_s u\|_p^p \mathrm{d}s$$

We shall also need the following result which give conditions of Malliavin differentiability of an Itô integral.

Theorem 2.3. *Let $u \in L^2(\Omega \times [0,T])$ be adapted and $X(t) = \int_0^t u(s)\,\mathrm{d}B(s),\ 0 \leq t \leq T$. Then we have*

$$u \in \mathbb{L}^{1,2} \text{ if and only if } X(t) \in \mathbb{D}^{1,2} \text{ for all } t \in [0,T].$$

We have in this case $X \in \mathbb{L}^{1,2}$ and for $0 \leq s \leq t \leq T$

$$D_s X(t) = u(s) 1_{[0,t]}(s) + \int_s^T D_s u(r)\,\mathrm{d}B(r).$$

Proof. See [2, 5, 6]. □

3. Main results

In this section, we give the main results of the paper. We shall need the subsequent existence, uniqueness and estimate result.

Theorem 3.1. *Let $\alpha \geq 1$ and set $p = 2\alpha$. Suppose that condition of Assumption A1 and Assumption A2 are satisfied. Then there exists a unique progressively measurable process*

$(Y(t), Z(t))$ solution of (2.1)-(2.2) in $S^p(0,T+\delta;\mathbb{R}) \times \mathbb{H}^p(0,T+\delta;\mathbb{R})$. Moreover, the solution of the time-advanced BSDE (2.1)-(2.2) satisfies

$$\begin{aligned} &E\Big[\sup_{0\le t\le T}|Y(t)|^p + \Big\{\int_0^T |Z(t)|^2 \mathrm{d}t\Big\}^{p/2}\Big] \\ &\le K_{T,p,\delta} E\Big[|G(T)|^p + \Big\{\int_T^{T+\delta} |G(t)|^2 \mathrm{d}t\Big\}^{p/2} + \Big\{\int_T^{T+\delta} |G_1(t)|^2 \mathrm{d}t\Big\}^{p/2} \\ &\quad + \Big\{\int_0^T |b(t,0,0,0,0,0,0,0,0,0)|\,\mathrm{d}t\Big\}^p\Big] \end{aligned} \tag{3.1}$$

Remark.

(1) The case $\alpha = 1$ is solved in [7].

(2) For $\alpha > 1$, the proof of existence and uniqueness follows in the same way as in [8, Theorem 5.3]. Note however that, in [8, Theorem 5.3], the norm in V_δ^p is replaced by the supremum norm, but this does not affect the proof in our case. We shall only prove that we can estimate the norm of the solution (Y,Z) by the parameters (G, G_1, b).

Before we give the main theorem of this paper, we shall suppose the following

Assumption A3.

(g2) $G \in \mathbb{D}^{1,2} \cap S^4(T,T+\delta;\mathbb{R}) \cap H^4(T,T+\delta;\mathbb{R})$, $G_1 \in \mathbb{D}^{1,2} \cap H^4(T,T+\delta;\mathbb{R})$ and

$$E\Big[\int_0^T |D_s G(T)|^2 \mathrm{d}s\Big] < \infty,\ E\Big[\int_0^T \int_T^{T+\delta} |D_s G(t)|^2 \mathrm{d}t\,\mathrm{d}s\Big] < \infty,$$

$$E\Big[\int_0^T \int_T^{T+\delta} |D_s G_1(t)|^2 \mathrm{d}t\,\mathrm{d}s\Big] < \infty$$

(b3) $b : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ is continuously differentiable in $(y, y_1, y_2, z, z_1, z_2)$ with uniformly bounded and continuous parttial derivatives $b_y, b_{y_1}, b_{y_2}, b_z, b_{z_1}, b_{z_2}$; and we set $b_y(t,\omega) = b_{y_1}(t,\omega) = b_{y_2}(t,\omega) = b_z(t,\omega) = b_{z_1}(t,\omega) = b_{z_2}(t,\omega) = 0$ for $\omega \in \Omega$, $t > T$

(b4) For each $(y, y_1, y_2, z, z_1, z_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, we have

$$b(\cdot, y, y_1, y_2, z, z_1, z_2) \in \mathbb{L}^{1,2},\ \ b(t,0,0,0,0,0,0) \in H^4(0,T;\mathbb{R}),$$

(b5) For $t \in [0,T]$ and $(y, y_1, y_2, z, z_1, z_2, \bar{y}, \bar{y}_1, \bar{y}_2, \bar{z}, \bar{z}_1, \bar{z}_2) \in \mathbb{R}^{12}$, we have

$$\begin{aligned} &\Big|D_s b(t, y, y_1, y_2, z, z_1, z_2) - D_s b(t, \bar{y}, \bar{y}_1, \bar{y}_2, \bar{z}, \bar{z}_1, \bar{z}_2)\Big| \\ &\le K_s(t)\Big(|y-\bar{y}| + |y_1 - \bar{y}_1| + |y_2 - \bar{y}_2|_{V_\delta^p} + |z - \bar{z}| + |z_1 - \bar{z}_1| + |z_2 - \bar{z}_2|_{V_\delta^p}\Big), \end{aligned}$$

where $(K_s(t))_{0\le s\le t}$ is a real valued measurable process which is $\mathcal{F}_t$-adapted in t and satisfies $\int_0^T \|K_s\|_4^4 \mathrm{d}s < \infty$.

(b6) For the unique solution (Y,Z) of the time-advanced BSDE (2.1)-(2.2) we suppose that

$$\int_0^T \|D_s b(t,Y,Y_1,Y_2,Z,Z_1,Z_2)\|_2^2 \mathrm{d}s < \infty$$

We are now ready to give our main theorem

Theorem 3.2. *Assume that (b,G,G_1) satisfies conditions in Assumptions A1-A3 (for $\alpha = 2$). Then $(Y,Z) \in \mathbb{L}^{1,2}(\mathbb{R}) \times \mathbb{L}^{1,2}(\mathbb{R})$, and a version of $(D_sY(t), D_sZ(t))_{0\le s,t\le T}$ is given by*

$$D_sY(t) = D_sZ(t) = 0,\ \ 0 \le t \le s \le T \tag{3.2}$$

$$dD_sY(t) = E\Big[D_sb(t)\Big|\mathcal{F}_t\Big]\mathrm{d}t + D_sZ(t)\,\mathrm{d}B(t),\ \ t \in [s,T] \tag{3.3}$$

$$D_sY(t) = D_sG(t),\ \ D_sZ(t) = D_sG_1(t),\ \ T \le s \le t \le T+\delta, \tag{3.4}$$

where

$$\begin{aligned} D_sb(t) = {} & \partial_y b(t) D_sY(t) + \partial_{y_1} b(t) D_sY(t+\delta) + \partial_{y_2} b(t) D_sY_t + \partial_z b(t) D_sZ(t) \\ & + \partial_{z_1} b(t) D_sZ(t+\delta) + \partial_{z_2} b(t) D_sZ_t + D_sb(t) \end{aligned}$$

with $b(t) = b(t, Y(t), Y(t+\delta), Y_t, Z(t), Z(t+\delta), Z_t)$.

Moreover, $(D_tY(t))_{0\le t\le T}$ is a version of $(Z(t))_{0\le t\le T}$

4. Proof of the theorems

In this Section, we prove our main theorems. We shall first prove estmate (3.1) .

Proof of the estimate (3.1). Let $t \in [0,T]$, applying Itô formula, we get

$$|Y(t)|^2 = |G(T)|^2 - 2\int_t^T E\Big[b(s)\Big|\mathcal{F}_s\Big]Y(s)\mathrm{d}s - \int_t^T |Z(s)|^2\mathrm{d}s - \int_t^T Y(s)Z(s)\,\mathrm{d}B(s) \tag{4.1}$$

where we have used the short hand notation

$$b(t) = b(t, Y(t), Y(t+\delta), Y_t, Z(t), Z(t+\delta), Z_t).$$

Take the conditional expectation with respect to $\mathcal{F}_t$, we get by averaging, using Lipschitz condition (Assumption A1) and Jensen inequality

$$
\begin{aligned}
&|Y(t)|^2 + E\Big[\int_t^T |Z(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \\
&\le E\Big[|G(T)|^2\Big|\mathcal{F}_t\Big] - 2E\Big[\int_t^T E\Big[b(s)\Big|\mathcal{F}_s\Big] Y(s)\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&\le E\Big[|G(T)|^2\Big|\mathcal{F}_t\Big] + 2C\Big\{E\Big[\int_t^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T |Y(s)|E\Big[|Y_1(s)|\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&+ E\Big[\int_t^T |Y(s)|E\Big[|Y_2(s)|_{V_\delta^p}\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T |Y(s)||Z(s)|\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&+ E\Big[\int_t^T |Y(s)|E\Big[|Z_1(s)|\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T |Y(s)|E\Big[|Z_2(s)|_{V_\delta^p}\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big]\Big\} \\
&+ 2E\Big[\int_t^T |b(s,0,0,0,0,0,0)Y(s)|\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&\le E\Big[|G(T)|^2\Big|\mathcal{F}_t\Big] + CE\Big[\int_t^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] + \frac{1}{\varepsilon_1}CE\Big[\int_t^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&+ \varepsilon_1 CE\Big[\int_t^T E\Big[|Y_1(s)|^2\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] + \frac{1}{\varepsilon_2}CE\Big[\int_t^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&+ \varepsilon_2 CE\Big[\int_t^T E\Big[|Y_2(s)|^2_{V_\delta^p}\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] + \frac{1}{\varepsilon_3}CE\Big[\int_t^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&+ \varepsilon_3 CE\Big[\int_t^T |Z(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] + \frac{1}{\varepsilon_4}CE\Big[\int_t^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&+ \varepsilon_4 CE\Big[\int_t^T E\Big[|Z_1(s)|^2\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] + \frac{1}{\varepsilon_5}CE\Big[\int_t^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&+ \varepsilon_5 CE\Big[\int_t^T E\Big[|Z_2(s)|^2_{V_\delta^p}\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] + 2E\Big[\int_t^T |b(s,0,0,0,0,0,0)Y(s)|\mathrm{d}s\Big|\mathcal{F}_t\Big]. \qquad (4.2)
\end{aligned}
$$

We also have that

$$
\begin{aligned}
E\Big[\int_t^T E\Big[|Y_1(s)|^2\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] &= E\Big[\int_t^T E\Big[|Y(s+\delta)|^2\Big|\mathcal{F}_s\Big]\,\mathrm{d}s\Big|\mathcal{F}_t\Big] \\
&\le E\Big[\int_t^T |Y(s)|^2\,\mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_T^{T+\delta} |G(s)|_H^2\,\mathrm{d}s\Big|\mathcal{F}_t\Big] \qquad (4.3)
\end{aligned}
$$

and interchanging the order of integration, we have

$$
E\Big[\int_t^T E\Big[|Y_2(s)|^2_{V_\delta^p}\Big|\mathcal{F}_s\Big]\mathrm{d}s\Big|\mathcal{F}_t\Big] \le \delta\Big(E\Big[\int_t^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_T^{T+\delta} |G(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big]\Big), \qquad (4.4)
$$

and similarly for Z. Therefore, (4.2) becomes

$$
\begin{aligned}
&|Y(t)|^2 + E\Big[\int_t^T |Z(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \\
&\leq E\Big[|G(T)|^2 \Big| \mathcal{F}_t\Big] + C^0_{1,\delta,\varepsilon} E\Big[\int_T^{T+\delta} |G(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] + C^0_{2,\delta,\varepsilon} E\Big[\int_T^{T+\delta} |G_1(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \\
&+ C_{1,\delta,\varepsilon} E\Big[\int_t^T |Y(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] + C_{2,\delta,\varepsilon} E\Big[\int_t^T |Z(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \\
&+ 2E\Big[\int_t^T |b(s,0,0,0,0,0,0)Y(s)| \mathrm{d}s \Big| \mathcal{F}_t\Big], \qquad (4.5)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{1,\delta,\varepsilon} &= C\Big(1 + \varepsilon_1 + \delta\varepsilon_2 + \sum_{i=1}^{5} \frac{1}{\varepsilon_i}\Big), \;\; C^0_{1,\delta,\varepsilon} = C(\varepsilon_1 + \delta\varepsilon_2), \\
C_{2,\delta,\varepsilon} &= C(\varepsilon_3 + \varepsilon_4 + \delta\varepsilon_5), \;\; C^1_{1,\delta,\varepsilon} = C(\varepsilon_4 + \delta\varepsilon_5).
\end{aligned}
$$

Choosing ε_3, ε_4 and ε_5 such that $C_{2,\delta,\varepsilon} < 1$, we get

$$
\begin{aligned}
&|Y(t)|^2 \\
&\leq E\Big[|G(T)|^2 \Big| \mathcal{F}_t\Big] + C^0_{1,\delta,\varepsilon} E\Big[\int_T^{T+\delta} |G(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] + C^0_{2,\delta,\varepsilon} E\Big[\int_T^{T+\delta} |G_1(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \\
&+ C_{1,\delta,\varepsilon} E\Big[\int_t^T |Y(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] + 2E\Big[\int_t^T |b(s,0,0,0,0,0,0)Y(s)| \mathrm{d}s \Big| \mathcal{F}_t\Big]. \qquad (4.6)
\end{aligned}
$$

Take the supremum leads to

$$
\begin{aligned}
&\sup_{u\leq t\leq T} |Y(t)|^2 \\
&\leq C_{1,\delta,\varepsilon} \sup_{u\leq t\leq T} E\Big[\int_u^T |Y(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] + \sup_{u\leq t\leq T} E\Big[|G(T)|^2 \Big| \mathcal{F}_t\Big] \\
&+ C^0_{1,\delta,\varepsilon} \sup_{u\leq t\leq T} E\Big[\int_T^{T+\delta} |G(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] + C^0_{2,\delta,\varepsilon} \sup_{u\leq t\leq T} E\Big[\int_T^{T+\delta} |G_1(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \\
&+ 2 \sup_{u\leq t\leq T} E\Big[\int_u^T |b(s,0,0,0,0,0,0)Y(s)| \mathrm{d}s \Big| \mathcal{F}_t\Big]. \qquad (4.7)
\end{aligned}
$$

Since

$$
\begin{aligned}
&E\Big[\int_u^T |Y(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big], \;\; E\Big[|G(T)|^2 \Big| \mathcal{F}_t\Big], \;\; E\Big[\int_T^{T+\delta} |G(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \\
&E\Big[\int_T^{T+\delta} |G_1(s)|^2 \mathrm{d}s \Big| \mathcal{F}_t\Big] \text{ and } E\Big[\int_u^T |b(s,0,0,0,0,0,0)Y(s)| \mathrm{d}s \Big| \mathcal{F}_t\Big]
\end{aligned}
$$

are right continuous semimartingales on $[0,T]$, with terminal values

$$E\Big[\int_u^T |Y(s)|^2\mathrm{d}s\Big],\ \ E\Big[|G(T)|^2\Big],\ \ E\Big[\int_T^{T+\delta} |G(s)|^2\mathrm{d}s\Big]$$
$$E\Big[\int_T^{T+\delta} |G_1(s)|^2\mathrm{d}s\Big] \text{ and } E\Big[\int_u^T |b(s,0,0,0,0,0,0)Y(s)|\mathrm{d}s\Big]$$

respectively. It then follows from the *Doob's martingale inequality* that, for $\alpha>1$

$$\begin{aligned} E\Big[\Big(\sup_{u\le t\le T} E\Big[\int_u^T |Y(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big]\Big)^\alpha\Big] &\le C_\alpha E\Big[\Big(\int_u^T |Y(s)|^2\mathrm{d}s\Big)^\alpha\Big]\\ &\le C_{1,\alpha,T}E\Big[\int_u^T \sup_{s\le r\le T}|Y(r)|^{2\alpha}\mathrm{d}s\Big], \end{aligned} \tag{4.8}$$

$$E\Big[\Big(\sup_{u\le t\le T} E\Big[|G(T)|^2\Big|\mathcal{F}_t\Big]\Big)^\alpha\Big] \le C_\alpha E\Big[|G(T)|^{2\alpha}\Big], \tag{4.9}$$

$$E\Big[\Big(\sup_{u\le t\le T} E\Big[\int_T^{T+\delta} |G(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big]\Big)^\alpha\Big] \le C_\alpha E\Big[\Big(\int_T^{T+\delta} |G(s)|^2\mathrm{d}s\Big)^\alpha\Big], \tag{4.10}$$

$$E\Big[\Big(\sup_{u\le t\le T} E\Big[\int_T^{T+\delta} |G_1(s)|^2\mathrm{d}s\Big|\mathcal{F}_t\Big]\Big)^\alpha\Big] \le C_\alpha E\Big[\Big(\int_T^{T+\delta} |G_1(s)|^2\mathrm{d}s\Big)^\alpha\Big], \tag{4.11}$$

and

$$\begin{aligned} &E\Big[\Big(\sup_{u\le t\le T} E\Big[\int_u^T |b(s,0,0,0,0,0,0)Y(s)|\mathrm{d}s\Big|\mathcal{F}_t\Big)^\alpha\Big]\\ &\le C_\alpha E\Big[\Big(\int_u^T |b(s,0,0,0,0,0,0)Y(s)|\mathrm{d}s\Big)^\alpha\Big]\\ &\le C_\alpha E\Big[\Big(\sup_{u\le t\le T}|Y(s)|^\alpha\Big)\Big(\int_u^T |b(s,0,0,0,0,0,0)|\mathrm{d}s\Big)^\alpha\Big]\\ &\le \varepsilon_6 C_\alpha E\Big[\sup_{u\le t\le T}|Y(s)|^{2\alpha}\Big] + \frac{1}{\varepsilon_6}C_\alpha E\Big[\Big(\int_u^T |b(s,0,0,0,0,0,0)|\mathrm{d}s\Big)^{2\alpha}\Big]. \end{aligned} \tag{4.12}$$

Combining (4.8)-(4.12) and choosing ε_6 such that $\varepsilon_6 C_\alpha<1$, we can find a constant K depending on α,δ,T and $\varepsilon_i,\ i=1,\dots,5$ and which can change from line line such that (4.7) becomes

$$\begin{aligned} &E\Big[\sup_{u\le t\le T}|Y(t)|^{2\alpha}\Big]\\ &\le C_{1,\delta,\varepsilon,\alpha,T}E\Big[\int_u^T \sup_{s\le r\le T}|Y(r)|^{2\alpha}\mathrm{d}s\Big] + KE\Big[|G(T)|^{2\alpha} + \Big(\int_T^{T+\delta}|G(r)|^2\mathrm{d}r\Big)^\alpha\\ &+\Big(\int_T^{T+\delta}|G_1(r)|^2\mathrm{d}r\Big)^\alpha + \Big(\int_u^T |b(r,0,0,0,0,0,0)|\mathrm{d}r\Big)^{2\alpha}\Big]. \end{aligned} \tag{4.13}$$

140 We then have from (4.13) that

$$
\begin{aligned}
&-\frac{d}{du}\Big(e^{Cu}E\Big[\int_u^T \sup_{s\le r\le T}|Y(r)|^{2\alpha}\mathrm{d}s\Big]\Big)\\
&\le Ke^{Cu}E\Big[|G(T)|^{2\alpha}+\Big(\int_T^{T+\delta}|G(r)|^2\mathrm{d}s\Big)^{\alpha}\\
&+\Big(\int_T^{T+\delta}|G_1(r)|^2\mathrm{d}r\Big)^{\alpha}+\Big(\int_u^T|b(r,0,0,0,0,0,0)|\mathrm{d}r\Big)^{2\alpha}\Big],
\end{aligned}
\tag{4.14}
$$

141 where $C=C_{1,\delta,\varepsilon,\alpha,T}$. Integrating from s to T, we get

$$
\begin{aligned}
&e^{Cs}E\Big[\int_s^T \sup_{t\le r\le T}|Y(r)|^{2\alpha}\mathrm{d}t\Big]\Big)\\
&\le K\int_s^T e^{Cu}E\Big[|G(T)|^{2\alpha}+\Big(\int_T^{T+\delta}|G(r)|^2\mathrm{d}r\Big)^{\alpha}\\
&+\Big(\int_T^{T+\delta}|G_1(r)|^2\mathrm{d}r\Big)^{\alpha}+\Big(\int_u^T|b(r,0,0,0,0,0,0)|\mathrm{d}r\Big)^{2\alpha}\Big]\,\mathrm{d}u,
\end{aligned}
\tag{4.15}
$$

142 This implies

$$
\begin{aligned}
&E\Big[\int_s^T \sup_{t\le r\le T}|Y(r)|^{2\alpha}\mathrm{d}t\Big]\Big)\\
&\le KE\Big[|G(T)|^{2\alpha}+\Big(\int_T^{T+\delta}|G(r)|^2\mathrm{d}r\Big)^{\alpha}\\
&+\Big(\int_T^{T+\delta}|G_1(r)|^2\mathrm{d}r\Big)^{\alpha}+\Big(\int_s^T|b(r,0,0,0,0,0,0)|\mathrm{d}r\Big)^{2\alpha}\Big].
\end{aligned}
\tag{4.16}
$$

143 Putting (4.13) and (4.16) together leads to

$$
\begin{aligned}
E\Big[\sup_{0\le t\le T}|Y(t)|^{2\alpha}\Big]&\le KE\Big[|G(T)|^{2\alpha}+\Big(\int_T^{T+\delta}|G(r)|^2\mathrm{d}r\Big)^{\alpha}\\
&+\Big(\int_T^{T+\delta}|G_1(r)|^2\mathrm{d}r\Big)^{\alpha}+\Big(\int_0^T|b(r,0,0,0,0,0,0)|\mathrm{d}r\Big)^{2\alpha}\Big].
\end{aligned}
\tag{4.17}
$$

144 On the other hand, since $\alpha>1$, we have by Burkholder-Davis-Gundy that

$$
E\Big[\Big(\int_0^T|Z(s)|^2\,\mathrm{d}s\Big)^{\alpha}\Big]\le C_\alpha E\Big[\Big|\int_0^T Z(s)\,\mathrm{d}B(s)\Big|^{2\alpha}\Big].
\tag{4.18}
$$

But

$$
\int_0^T Z(s)\,\mathrm{d}B(s)=G(T)+\int_0^T E\Big[b(t,Y(t),Y(t+\delta),Y_t,Z(t),Z(t+\delta),Z_t)\Big|\mathcal{F}_t\Big]\mathrm{d}t-Y(0)
$$

It follows that

$$E\Big[\Big(\int_0^T |Z(s)|^2\,\mathrm{d}s\Big)^{\alpha}\Big] \le C_\alpha E\Big[|G(T)|^{2\alpha} + |Y(0)|^{2\alpha} + \Big(\int_0^T E\Big[|b(t,Y(t),Y(t+\delta),Y_t,Z(t),Z(t+\delta),Z_t)|\Big|\mathcal{F}_t\Big]\mathrm{d}t\Big)^{\alpha}\Big] \tag{4.19}$$

By averaging over $Y(t), Y(t+\delta), Y_t, Z(t), Z(t+\delta), Z_t$, using Lipschitz condition, inequalities of type (4.3), (4.4) and the estimate (4.17), we get

$$E\Big[\Big(\int_0^T |Z(s)|^2\,\mathrm{d}s\Big)^{\alpha}\Big] \le KE\Big[|G(T)|^{2\alpha} + \Big(\int_T^{T+\delta} |G(r)|^2\mathrm{d}r\Big)^{\alpha} + \Big(\int_T^{T+\delta} |G_1(r)|^2\mathrm{d}r\Big)^{\alpha} + \Big(\int_0^T |b(r,0,0,0,0,0,0)|\mathrm{d}r\Big)^{2\alpha}\Big]. \tag{4.20}$$

The estimate (3.1) then follows by combining (4.17) and (4.20).

□

We shall prove Theorem 3.2 in three steps. The arguments are mainly based upon the estimate (3.1).

We shall first define in a recursive way, approximations of the solution of the time-advanced BSDE (2.1)-(2.2).

For $n \ge 0$, define

$$Y^0 = Z^0 = 0,$$

$$Y^{n+1}(t) = G(T) - \int_t^T E\Big[b^n(r)\Big|\mathcal{F}_r\Big]\mathrm{d}r - \int_t^T Z^{n+1}(r)\,\mathrm{d}B(r),\ t\in[0,T]; \tag{4.21}$$

$$Y^{n+1}(t) = G(t),\ \ Z^{n+1}(t) = G_1(t),\ \ t\in[T,T+\delta], \tag{4.22}$$

where $b^n(t) = b(t, Y^n(t), Y^n(t+\delta), Y^n_t, Z^n(t), Z^n(t+\delta), Z^n_t)$.

Lemma 4.1. *Under conditions of Theorem 3.2, the sequence (Y^n, Z^n) converges in $S^4(0,T+\delta;\mathbb{R})\times\mathbb{H}^4(0,T+\delta;\mathbb{R})$ to (Y,Z) as $n\to\infty$.*

Proof. Combining arguments used in the proof of [8, Theorem 5.3] and in the proof of Theorem 3.1, we can easily show that the sequence (Y^n, Z^n) is a Cauchy sequence in the complete space $S^4(0,T+\delta;\mathbb{R})\times\mathbb{H}^4(0,T+\delta;\mathbb{R})$. We can then deduce that (Y^n,Z^n) converges to some limit (Y,Z) which is the unique solution of (2.1)-(2.2). □

Lemma 4.2. *Assume that (b,G,G_1) satisfies conditions of Theorem 3.2. Then the sequence (Y^n,Z^n) defined by (4.21)-(4.22) belongs to $\mathbb{L}^{1,2}(\mathbb{R})\times\mathbb{L}^{1,2}(\mathbb{R})$*

Proof. We shall prove it by recursion. The statement is true for $n=0$. Let it be true for n.

Step 1. We shall prove that $\int_t^T E\Big[b^n(r)\Big|\mathcal{F}_r\Big]\mathrm{d}r$ is Malliavin differentiable. Notice that $Y^n(t) \in \mathbb{D}^{1,2}$ for λ-a.e. $t \in [0, T+\delta]$ and from our hypothesis, we have

$$\begin{aligned}\int_0^T E\Big[|Y^n(r+\delta)|^2\Big]\mathrm{d}r &\leq E\Big[\int_\delta^{T+\delta}|Y^n(r)|^2\mathrm{d}r\Big]\\&\leq C_T E\Big[\sup_{0\leq r\leq T}|Y^n(r)|^2\Big] + E\Big[\int_T^{T+\delta}|G^n(r)|^2\mathrm{d}r\Big] < \infty,\end{aligned}\tag{4.23}$$

and

$$\begin{aligned}&E\Big[\int_0^T\Big(\int_0^T|D_sY^n(r+\delta)|^2\mathrm{d}r\Big)\,\mathrm{d}s\Big]\\&\leq E\Big[\int_0^T\int_\delta^{T+\delta}|D_sY^n(r)|^2\mathrm{d}r\,\mathrm{d}s\Big]\\&\leq C_T E\Big[\int_0^T\sup_{0\leq r\leq T}|D_sY^n(r)|^2\mathrm{d}s\Big] + \int_0^T\int_T^E\Big[T+\delta|D_sG(r)|^2\mathrm{d}r\Big]\,\mathrm{d}s < \infty,\end{aligned}\tag{4.24}$$

and interchanging

$$\begin{aligned}\int_0^T E\Big[|Y_r^n|^2_{L^2([0,\delta],\mathrm{d}u)}\Big]\mathrm{d}r &\leq \int_0^T E\Big[\int_0^\delta|Y^n(r+u)|^2\mathrm{d}u\Big]\,\mathrm{d}r\\&\leq \int_0^T E\Big[\int_r^{r+\delta}|Y^n(u)|^2\mathrm{d}u\Big]\,\mathrm{d}r\\&\leq C_{\delta,T}E\Big[\sup_{0\leq r\leq T}|Y^n(r)|^2\Big] + \delta E\Big[\int_T^{T+\delta}|G(r)|^2\mathrm{d}r\Big] < \infty,\end{aligned}\tag{4.25}$$

and

$$\begin{aligned}&E\Big[\int_0^T\int_0^T|D_sY_r^n|^2_{L^2([0,\delta],\mathrm{d}u)}\mathrm{d}r\,\mathrm{d}s\Big]\\&= E\Big[\int_0^T\int_0^T\Big(\int_0^\delta|D_sY^n(r+u)|^2\mathrm{d}u\Big)\mathrm{d}r\,\mathrm{d}s\Big]\\&\leq \int_0^T E\Big[\int_0^T\int_r^{r+\delta}|D_sY^n(u)|^2\mathrm{d}u\,\mathrm{d}r\Big]\mathrm{d}s\\&\leq C_{\delta,T}E\Big[\int_0^T\sup_{0\leq r\leq T}|D_sY^n(r)|^2\mathrm{d}s\Big] + \delta\int_0^T E\Big[\int_T^{T+\delta}|D_sG(r)|^2\mathrm{d}r\Big]\,\mathrm{d}s < \infty.\end{aligned}\tag{4.26}$$

It follows from these inequalities that $Y^n(t+\delta)$, $Y_t^n \in \mathbb{D}^{1,2}$. Similarly we show that $Z^n(t+\delta)$, $Z_t^n \in \mathbb{D}^{1,2}$

We claim that $\int_0^T E\Big[|b^n(r)|^2\Big]\mathrm{d}r < \infty$.

This follows by averaging over $Y^n(t), Y^n(t+\delta), Y_t^n, Z^n(t), Z^n(t+\delta), Z_t^n$, using **(g1), (b1), (b2)** and the fact that $(Y^n, Z^n) \in S^4 \times \mathbb{H}^4$.

We also claim that

$$\int_0^T E\Big[\Big(\int_0^T \Big|D_s E\Big[b^n(r)\Big|\mathcal{F}_r\Big]\Big|\mathrm{d}r\Big)^2\Big]\mathrm{d}s < \infty$$

.

This follows once more by averaging over $Y^n(t), Y^n(t+\delta), Y_t^n, Z^n(t), Z^n(t+\delta), Z_t^n$, using the fact that $(Y^n, Z^n) \in S^4 \times \mathbb{H}^4$ and applying **(b4)** and **(b5)**.

We can then deduce from the preceding inequalities that $\int_t^T E[b^n(r)|\mathcal{F}_r]\mathrm{d}r$ is Malliavin differentiable. Therefore for $0 \le t \le T$

$$G(T) - \int_t^T E\Big[b^n(r)\Big|\mathcal{F}_r\Big]\mathrm{d}r \in \mathbb{D}^{1,2}$$

with Malliavin derivative

$$D_s G(T) - \int_t^T E\Big[D_s b^n(r)\Big|\mathcal{F}_r\Big]1_{[0,r]}(s)\,\mathrm{d}r \tag{4.27}$$

where $D_s b^n$ is define by

$$\begin{aligned} D_s b^n(t) = \partial_y b^n(t) D_s Y^n(t) + \partial_{y_1} b^n(t) D_s Y^n(t+\delta) + \partial_{y_2} b^n(t) D_s Y_t^n + \partial_z b^n(t) D_s Z^n(t) \\ + \partial_{z_1} b^n(t) D_s Z^n(t+\delta) + \partial_{z_2} b^n(t) D_s Z_t^n + D_s b^n(t) \end{aligned} \tag{4.28}$$

Step 2. We shall prove that $(Y^{n+1}, Z^{n+1}) \in \mathbb{D}^{1,2} \times \mathbb{D}^{1,2}$.
Since

$$Y^{n+1}(t) = E\Big[G(T) - \int_t^T E\Big[b^n(r)\Big|\mathcal{F}_r\Big]\mathrm{d}r\Big|\mathcal{F}_t\Big]$$

We deduce that $Y^{n+1} \in \mathbb{D}^{1,2}$ and from (4.21) it follows that $\int_t^T Z^{n+1}(r)\,\mathrm{d}B(r) \in \mathbb{D}^{1,2}$. Theorem 2.3 implies that $Z^{n+1} \in \mathbb{L}^{1,2}$ and we have

$$\begin{aligned} D_s \int_t^T Z^{n+1}(r)\,\mathrm{d}B(r) &= \int_t^T D_s Z^{n+1}(r)\,\mathrm{d}B(r), \quad s \le t \\ D_s \int_t^T Z^{n+1}(r)\,\mathrm{d}B(r) &= Z^{n+1}(s) + \int_s^T D_s Z^{n+1}(r)\,\mathrm{d}B(r), \quad s > t. \end{aligned}$$

We can then differentiate (4.21) and obtain

$$\begin{aligned} D_s Y^{n+1}(t) = D_s G(T) &- \int_t^T E\Big[D_s b^n(r)\Big|\mathcal{F}_r\Big]1_{[0,r]}(s)\,\mathrm{d}r \\ &- \int_t^T D_s Z^{n+1}(r)\,\mathrm{d}B(r), \quad s \le t \le T \end{aligned} \tag{4.29}$$

$$D_s Y^{n+1}(t) = D_s G(t), \; D_s Z^{n+1}(t) = D_s G_1(t), \;\; T \le s \le t \le T+\delta \tag{4.30}$$

$$D_s Y^{n+1}(t) = D_s Z^{n+1}(t) = 0 \;\; t < s. \tag{4.31}$$

The time-advanced BSDE (4.29)-(4.30) with $D_s b^n(t)$ given by (4.28) satisfies the conditions of Theorem 3.1 since **(g2)** and **(b3)** hold. We deduce that there exists a unique solution $(D_s Y^{n+1}, D_s Z^{n+1}) \in S^2 \times \mathbb{H}^2$ of (4.29)-(4.30) satisfying (4.31).

It follows in the same way as in the proof of the estimate (3.1) that there exists a constant K such that

$$\|D_sY^{n+1}\|^2_{S^2} + \|D_sZ^{n+1}\|^2_{\mathbb{H}^2} \leq KE\Big[|D_sG(T)|^2 + \int_T^{T+\delta} |D_sG(t)|^2 \mathrm{d}t + \int_T^{T+\delta} |D_sG_1(t)|^2 \mathrm{d}t \\ + \|D_sb^n(\cdot, Y^n(t), Y^n(t+\delta), Y^n_t, Z^n(t), Z^n(t+\delta), Z^n_t)\|^2_2\Big] \\ + \|D_sY^n\|^2_{S^2} + \|D_sZ^n\|^2_{\mathbb{H}^2}$$

This implies that $E\Big[\int_0^T \sup\limits_{0\leq t\leq T} |D_sY^{n+1}(t)|\mathrm{d}s\Big] < \infty$ and then $Y^{n+1} \in \mathbb{L}^{1,2}$.

□

Lemma 4.3. *The sequence* (D_sY^n, D_sZ^n) *converges to* (Y^s, Z^s) *in* $L^2(\Omega \times [0,T]^2; S^2 \times \mathbb{H}^2)$ *where for* $0 \leq s \leq T + \delta$, (Y^s, Z^s) *is the solution of the time-advanced BSDE*

$$Y^s(t) = D_sG(T) - \int_t^T E\Big[b^s(r)\Big|\mathcal{F}_r\Big] 1_{[0,r]}(s)\,\mathrm{d}r \\ - \int_t^T Z^s(r)\,\mathrm{d}B(r), \;\; s \leq t \leq T \tag{4.32}$$

$$Y^s(t) = D_sG(t), \;\; Z^s(t) = D_sG_1(t), \;\; T \leq s \leq t \leq T + \delta \tag{4.33}$$

$$Y^s(t) = Z^s(t) = 0 \;\; t < s, \tag{4.34}$$

where

$$b^s(t) = \partial_y b(t)Y^s(t) + \partial_{y_1} b(t)Y^s(t+\delta) + \partial_{y_2} b(t)Y^s_t + \partial_z b(t)Z^s(t) \\ + \partial_{z_1} b(t)Z^s(t+\delta) + \partial_{z_2} b(t)Z^s_t + D_sb(t),$$

with $b(t) = b(t, Y(t), Y(t+\delta), Y_t, Z(t), Z(t+\delta), Z_t))$.

Proof. We shall first show that the solution (Y^s, Z^s) of the time-advanced BSDE (4.32)-(4.34) is integrable. Note that the generator b^s and the terminal values G and G_1) satisfy conditions of Theorem 3.1 therefore the estimate (4.31) holds.

It then follows that there exists a constant K such that

$$\|Y^s\|^2_{S^2} + \|Z^s\|^2_{\mathbb{H}^2} \leq KE\Big[|D_sG(T)|^2 + \int_T^{T+\delta} |D_sG(t)|^2 \mathrm{d}t + \int_T^{T+\delta} |D_sG_1(t)|^2 \mathrm{d}t \\ + \|D_sb(\cdot, Y(t), Y(t+\delta), Y_t, Z(t), Z(t+\delta), Z_t)\|^2_2\Big]$$

which implies that

$$\int_0^T \Big(\|Y^s\|^2_{S^2} + \|Z^s\|^2_{\mathbb{H}^2}\Big)\mathrm{d}s < \infty. \tag{4.35}$$

We next give an estimate for differences of (D_sY^n, D_sZ^n) and (Y^s, Z^s). Fix $n \in \mathbb{N}$ and consider the difference process $(D_sY^n - Y^s, D_sZ^n - Z^s)$ with $G = 0$ and $G_1 = 0$ as terminal conditions. Applying estimate (4.31) for $\alpha = 1$ to this process yields the following inequality

$$
\begin{aligned}
\|D_sY^n - Y^s\|_{S^2}^2 + \|D_sZ^n - Z^s\|_{\mathbb{H}^2}^2 &\le KE\Big[\Big(\int_s^T |\gamma^n(t)|\mathrm{d}t\Big)^2\Big] \\
&\le K\Big(I_{n-1}^{1,s}(T) + I_{n-1}^{2,s}(T) + I_{n-1}^{3,s}(T)\Big)
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma^n(t) = {}& D_sb(t) - D_sb^{n-1}(t) + \partial_y b(t)Y^s(t) - \partial_y b^{n-1}(t)D_sY^{n-1}(t) \\
&+ \partial_{y_1} b(t)Y^s(t+\delta) - \partial_{y_1} b^{n-1}(t)D_sY^{n-1}(t+\delta) + \partial_{y_2} b(t)Y_t^s - \partial_{y_2} b^{n-1}(t)D_sY_t^{n-1} \\
&+ \partial_z b(t)Z^s(t) - \partial_z b^{n-1}(t)D_sZ^{n-1}(t) + \partial_{z_1} b(t)Z^s(t+\delta) - \partial_{z_1} b^{n-1}(t)D_sZ^{n-1}(t+\delta) \\
&+ \partial_{z_2} b(t)Z_t^s - \partial_{z_2} b^{n-1}(t)D_sZ_t^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
I_{n-1}^{1,s}(T) = {}& E\Big[\Big(\int_s^T \Big|D_sb(t) - D_sb^{n-1}(t)\Big|\mathrm{d}t\Big)^2\Big] \\
I_{n-1}^{2,s}(T) = {}& E\Big[\Big(\int_s^T \Big|\partial_y b(t)\Big\{Y^s(t) - D_sY^{n-1}(t)\Big\}\Big|\mathrm{d}t\Big)^2 \\
&+ \Big(\int_s^T \Big|\partial_{y_1} b(t)\Big\{Y^s(t+\delta) - D_sY^{n-1}(t+\delta)\Big\}\Big|\mathrm{d}t\Big)^2 + \Big(\int_s^T \Big|\partial_{y_2} b(t)\Big\{Y_t^s - D_sY_t^{n-1}\Big\}\Big|\mathrm{d}t\Big)^2 \\
&+ \Big(\int_s^T \Big|\partial_z b(t)\Big\{Z^s(t) - D_sZ^{n-1}(t)\Big\}\Big|\mathrm{d}t\Big)^2 + \Big(\int_s^T \Big|\partial_{z_2} b(t)(Z_t^s - D_sZ_t^{n-1})\Big|\mathrm{d}t\Big)^2 \\
&+ \Big(\int_s^T \Big|\partial_{z_1} b(t)\Big\{Z^s(t+\delta) - D_sZ^{n-1}(t+\delta)\Big\}\Big|\mathrm{d}t\Big)^2\Big] \\
I_{n-1}^{3,s}(T) = {}& E\Big[\Big(\int_s^T \Big|\Big\{\partial_y b(t) - \partial_y b^{n-1}(t)\Big\}Y^s(t)\Big|\mathrm{d}t\Big)^2 + \Big(\int_s^T \Big|\Big\{\partial_{y_1} b(t) - \partial_{y_1} b^{n-1}(t)\Big\}Y^s(t+\delta)\Big|\mathrm{d}t\Big)^2 \\
&+ \Big(\int_s^T \Big|\Big\{\partial_{y_2} b(t) - \partial_{y_2} b^{n-1}(t)\Big\}Y_t^s\Big|\mathrm{d}t\Big)^2 + \Big(\int_s^T \Big|\Big\{\partial_z b(t) - \partial_y b^{n-1}(t)\Big\}Z^s(t)\Big|\mathrm{d}t\Big)^2 \\
&+ \Big(\int_s^T \Big|\Big\{\partial_{z_1} b(t) - \partial_{z_1} b^{n-1}(t)\Big\}Z^s(t+\delta)\Big|\mathrm{d}t\Big)^2 + \Big(\int_s^T \Big|\Big\{\partial_{z_2} b(t) - \partial_{z_2} b^{n-1}(t)\Big\}Z_t^s\Big|\mathrm{d}t\Big)^2\Big]
\end{aligned}
$$

Using condition **(b5)**, we get

$$
\begin{aligned}
& I_{n-1}^{1,s}(T) \\
& \leq E\Big[\Big(\int_s^T K_s(t)\Big\{|Y(t)-Y^{n-1}(t)|+|Y(t+\delta)-Y^{n-1}(t+\delta)| \\
& +|Y_t-Y_t^{n-1}|+|Z(t)-Z^{n-1}(t)|+|Z(t+\delta)-Z^{n-1}(t+\delta)|+|Z_t-Z_t^{n-1}|\Big\}\mathrm{d}t\Big)^2\Big] \\
& \leq C_\delta\Big(E\Big[\int_s^T K_s(t)^4\mathrm{d}t\Big]^{\frac{1}{2}}\Big)\Big(E\Big[\int_s^T |Y(t)-Y^{n-1}(t)|^4\mathrm{d}t\Big]^{\frac{1}{2}}+E\Big[\int_s^T |Z(t)-Z^{n-1}(t)|^4\mathrm{d}t\Big]^{\frac{1}{2}}\Big) \\
& \leq C_\delta\|K_s\|_4^2\Big(\|Y-Y^{n-1}\|_4^2+\|Z-Z^{n-1}\|_4^2\Big),
\end{aligned}
$$

where the second inequality follows from Cauchy-Schwartz and (4.3)-(4.4) with $G(T)=G_1(T)=0$. From Lemma 4.1, we have that (Y^n,Z^n) converges to (Y,Z) in $S^4\times\mathbb{H}^4$. Hence

$$
\lim_{n\to\infty}\int_0^T I_{n-1}^{1,s}(T)\,\mathrm{d}s=0.
$$

In addition, since the partial derivatives of b with respect to y,y_1,y_2,z,z_1,z_2 are bounded and continuous, and since the integrability condition (4.35) holds, we conclude using *dominated convergence theorem* that

$$
\lim_{n\to\infty}\int_0^T I_{n-1}^{3,s}(T)\,\mathrm{d}s=0.
$$

Finally we shall show that $I_{n-1}^{2,s}(T)$ converges to 0 as $n\to\infty$. Using once more the boundedness of the partials derivatives of b and (4.3)-(4.4), we get

$$
I_{n-1}^{2,s}(T)\leq C_{T,\delta}\Big(\|Y^s-D_sY^{n-1}\|_{S^2}^2+\|Z^s-D_sZ^{n-1}\|_{\mathbb{H}^2}^2\Big)
$$

Now choose T so that $\theta=C_{T,\delta}<1$. Let $\varepsilon>0$. Then there exists $N\in\mathbb{N}$ such that, for any $n\geq N$,

$$
\begin{aligned}
& E\Big[\int_0^T \|Y^s-D_sY^n\|_{S^2}^2+\|Z^s-D_sZ^n\|_{\mathbb{H}^2}^2\mathrm{d}s\Big] \\
& \leq \varepsilon+\theta E\Big[\int_0^T \|Y^s-D_sY^{n-1}\|_{S^2}^2+\|Z^s-D_sZ^{n-1}\|_{\mathbb{H}^2}^2\mathrm{d}s\Big].
\end{aligned}
$$

Iterating this inequality, we get

$$
\begin{aligned}
&E\Big[\int_0^T \|Y^s - D_sY^n\|^2_{S^2} + \|Z^s - D_sZ^n\|^2_{\mathbb{H}^2}\mathrm{d}s\Big] \\
&\leq \varepsilon(1+\theta+\theta^2+\ldots,\theta^{n-N-1}) + \theta^{n-N}E\Big[\int_0^T \|Y^s - D_sY^N\|^2_{S^2} + \|Z^s - D_sZ^N\|^2_{\mathbb{H}^2}\mathrm{d}s\Big] \\
&\leq \frac{\varepsilon}{1-\theta} + \theta^{n-N}E\Big[\int_0^T \|Y^s - D_sY^N\|^2_{S^2} + \|Z^s - D_sZ^N\|^2_{\mathbb{H}^2}\mathrm{d}s\Big] \\
&\leq \frac{\varepsilon}{1-\theta} + \theta^{n-N}K,
\end{aligned}
$$

where K is a positive constant. Now let $n \to \infty$, since ε is arbitrary and $0 \leq \theta < 1$, it follows that

$$\lim_{n\to\infty}\int_0^T I^{2,s}_{n-1}(T)\,\mathrm{d}s = 0.$$

We conclude that the sequence (D_sY^n, D_sZ^n) converges to (Y^s, Z^s) in $L^2(\Omega\times[0,T]^2; S^2\times\mathbb{H}^2)$.

□

Proof of Theorem 3.2. Since $\mathbb{L}^{1,2}$ is a Hilbert space and D is a closed operator, we conclude from the preceding lemmas that the unique solution (Y,Z) of the time-advanced BSDE (2.1)-(2.2) belongs to $\mathbb{L}^{1,2}\times\mathbb{L}^{1,2}$ and that $(Y^s,Z^s)_{0\leq s\leq T}$ is a version of $(D_sY, D_sZ)_{0\leq s\leq T}$.

It remains to show that $(D_sY(s))_{0\leq s\leq T}$ is a version of $(Z(s))_{0\leq s\leq T}$.

Note that for $t \leq s \leq T$

$$Y(t) = Y(s) - \int_t^s E\Big[b(r,Y(r),Y(r+\delta),Y_r,Z(r),Z(r+\delta),Z_r)\Big|\mathcal{F}_r\Big]\mathrm{d}r - \int_t^s Z(r)\,\mathrm{d}B(r).$$

It follows from Theorem 2.3, that for $t < u \leq s$

$$D_uY(t) = Z(u) - \int_u^s E\Big[D_ub(r)\Big|\mathcal{F}_r\Big]\mathrm{d}r - \int_u^s D_uZ(r)\,\mathrm{d}B(r),$$

with

$$
\begin{aligned}
D_ub(r) &= D_ub(r,Y(r),Y(r+\delta),Y_r,Z(r),Z(r+\delta),Z_r) \\
&= \partial_y b(r)D_uY(r) + \partial_{y_1}b(r)D_uY^n(r+\delta) + \partial_{y_2}b(r)D_uY_r + \partial_z b(r)D_uZ(r) \\
&\quad + \partial_{z_1}b(r)D_uZ(r+\delta) + \partial_{z_2}b(r)D_uZ_t + D_ub(r).
\end{aligned}
$$

Hence, taking $u = s$ leads to $D_sY(s) = Z(s)$ a.s. □

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CMA, Department of Mathematics, University of Oslo, Moltke Moes vei 35, P.O. Box 1053 Blindern, 0316 Oslo, Norway.
E-mail address: o.m.pamen@cma.uio.no